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Pion Mass and the PCAC Relation in the Overlap Fermion Formalism:
Gauged Gross-Neveu Model on a Lattice

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Abstract

We investigate chiral properties of the overlap lattice fermion by using solvable model in two dimensions, the gauged Gross-Neveu model. In this model, the chiral symmetry is spontaneously broken in the presence of small but finite fermion mass. We calculate the quasi-Nambu-Goldstone(NG) boson mass as a function of the bare fermion mass and two parameters in the overlap formula. We find that the quasi-NG boson mass has desired properties as a result of the extended chiral symmetry found by Lüscher. We also examine the PCAC relation and find that it is satisfied in the continuum limit. Comparison between the overlap and Wilson lattice fermions is made.

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Species doubling is a long standing problem in the lattice fermion formulation. Wilson fermion is the most suitable formulation[1] and it is used in most of the numerical studies of lattice gauge theory. However in order to reach the desired continuum limit, fine tunning must be done with respect to the "bare fermion mass" and the Wilson parameter.

Recently a very promising formulation of lattice fermion named overlap fermion was proposed by Narayanan and Neuberger[2]. In that formula the Ginsparg and Wilson(GW) relation[3] plays a very important role, and because of that there exists an "extended" (infinitesimal) chiral symmetry.

In this paper we shall study or test the overlap fermion by using the gauged Gross-Neveu model in two dimensions. This is a solvable model which has similar chiral properties with QCD_4, i.e., chiral symmetry is spontaneously broken with a small but finite bare fermion mass and pion appears as quasi-Nambu-Goldstone boson[1]. Actually a closely related model was studied on a lattice in order to test properties of the Wilson fermion in the continuum limit[5]. Therefore advantage of the overlap fermion becomes clear by the investigation in this paper.

The model is defined by the following action on the two-dimensional square lattice with the lattice spacing a,

$$\begin{aligned} S &= \frac{N}{2}\sum_{pl}\prod U_\mu(n) + a^2\sum_{n,m}\bar{\psi}(m)D(m,n)\psi(n) + a^2 M_B\sum_n \bar{\psi}\psi(n) \\ &\quad -\frac{a^2}{\sqrt{N}}\sum_n\Big[\phi^i(n)(\bar{\psi}\tau^i\psi)(n) + \phi_5^i(n)(\bar{\psi}\tau^i\gamma_5\psi)(n)\Big] \\ &\quad +\frac{a^2}{2g_v}\sum_n\Big[\phi^i(n)\phi^i(n) + \phi_5^i(n)\phi_5^i(n)\Big], \end{aligned} \tag{1}$$

where $U_\mu(n)$ is U(1) gauge field defined on links, ψ^l_α ($\alpha = 1, ..., N, l = 1, ..., L$) are fermion fields with flavour index l, and the matrix τ^i ($i = 0, ..., L^2 - 1$) acting on the flavour index is normalized as

$$\mathrm{Tr}(\tau^i\tau^k) = \delta_{ik} \tag{2}$$

[1]Overlap fermion with a finite fermion mass was recently studied in Ref.[4]

and

$$\tau^0 = \frac{1}{\sqrt{L}}, \ \ \{\tau^i, \tau^j\} = d^{ijk}\tau^k, \tag{3}$$

where d^{ijk}'s are the structure constants of $SU(L)$. Fields ϕ^i and ϕ_5^i are scalar and pseudo-scalar bosons,respectively. The covariant derivative in Eq.(1) is defined by the overlap formula

$$\begin{aligned} D &= \frac{1}{a}\Big(1 + X\frac{1}{\sqrt{X^\dagger X}}\Big), \\ X_{nm} &= \gamma_\mu C_\mu(n,m) + B(n,m), \\ C_\mu &= \frac{1}{2a}\Big[\delta_{m+\mu,n}U_\mu(m) - \delta_{m,n+\mu}U_\mu^\dagger(n)\Big], \\ B(n,m) &= -\frac{M_0}{a} + \frac{r}{2a}\sum_\mu\Big[2\delta_{n,m} - \delta_{m+\mu,n}U_\mu(m) - \delta_{m,n+\mu}U_\mu^\dagger(n)\Big], \end{aligned} \tag{4}$$

where r and M_0 are dimensionless nonvanishing free parameters of the overlap lattice fermion formalism[2, 6]. The overlap Dirac operator D does not have the ordinary chiral invariance but satisfies the GW relation instead,

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D. \tag{5}$$

From (1) it is obvious that the systematic $1/N$ expansion is possible and we shall employ it.

The action (1) contains the bare fermion mass M_B which explicitly breaks the chiral symmetry. This bare mass also breaks the following infinitesimal transformation, which was discovered by Lüscher[7] and we call "extended chiral symmetry",

$$\begin{aligned} &\psi(n) \to \psi(n) + \tau^k\theta^k\gamma_5\Big\{\delta_{nm} - \frac{1}{2}aD(n,m)\Big\}\psi(m), \\ &\bar\psi(n) \to \bar\psi(n) + \bar\psi(m)\tau^k\theta^k\gamma_5\Big\{\delta_{nm} - \frac{1}{2}aD(n,m)\Big\} \\ &\phi^i(n) \to \phi^i(n) + d^{ikj}\theta^k\phi_5^j(n), \\ &\phi_5^i(n) \to \phi_5^i(n) - d^{ikj}\theta^k\phi^j(n), \end{aligned} \tag{6}$$

where θ^i is an infinitesimal transformation parameter. The above symmetry (6) coincides with the ordinary chiral symmetry up to $O(a)$.

From the action (1), it is obvious that ϕ^i and ϕ_5^i are composite fields of the fermions,

$$\phi^i = \frac{g_v}{\sqrt{N}}\bar{\psi}\tau^i\psi, \quad \phi_5^i = \frac{g_v}{\sqrt{N}}\bar{\psi}\gamma_5\tau^i\psi. \tag{7}$$

As in the continuum model, we expect that the field ϕ^0 acquires a nonvanishing vacuum expectation value(VEV),

$$\langle\phi^0\rangle = \sqrt{NL}M_s, \tag{8}$$

and we define subtracted fields,

$$\begin{aligned}\varphi^0 &= \phi^0 - \sqrt{NL}M_s, \\ \varphi^i &= \phi^i \;\; (i \neq 0), \quad \varphi_5^i = \phi_5^i.\end{aligned} \tag{9}$$

In terms of the above fields,

$$\begin{aligned} S \;\; = \;\; & \frac{N}{2}\sum_{pl}\prod U_\mu(n) + a^2\sum_{n,m}\bar{\psi}(m)D_M(m,n)\psi(n) \\ & -\frac{a^2}{\sqrt{N}}\sum_n\Big[\varphi^i(n)(\bar{\psi}\tau^i\psi)(n) + \varphi_5^i(n)(\bar{\psi}\tau^i\gamma_5\psi)(n)\Big] \\ & +\frac{a^2}{2g_v}\sum_n\Big[\varphi^i(n)\varphi^i(n) + 2\sqrt{NL}M_s\varphi^0(n) + \varphi_5^i(n)\varphi_5^i(n)\Big], \end{aligned} \tag{10}$$

where

$$D_M = D - M, \quad M = M_B + M_s. \tag{11}$$

Obviously, M is the dymanical fermion mass.

From the chiral symmetry and (8), we can expect that quasi-Nambu-Goldstone(NG) bosons appear as a result of the spontaneous breaking of the chiral symmetry. They are nothing but φ_5^i. The VEV M_s is determined by the tadpole cancellation condition of φ^0. In order to perform an explicit calculation of the $1/N$-expansion, it is useful to employ the momentum representation, and also we introduce the gauge potential $\lambda_\mu(n)$ in the usual way, i.e., $U(n,\mu) = \exp(\frac{ia}{\sqrt{N}}\lambda_\mu(n))$. By using weak-coupling

expansion by Kikukawa and Yamada[8],

$$D_{nm} = \int_p \int_q e^{-ia(qn-pm)} D(p,q), \tag{12}$$

$$D(p,q) = D_0(p)(2\pi)^2 \delta(p-q) + \frac{1}{a} V(p,q), \tag{13}$$

where $\int_p = \int_{-\pi/a}^{\pi/a} \frac{d^2p}{(2\pi)^2}$ and

$$D_0(p) = \frac{b(p)+\omega(p)}{a\omega(p)} + \frac{\gamma_\mu i \sin ap_\mu}{a^2\omega(p)}, \tag{14}$$

$$V(p,q) = \Big\{\frac{1}{\omega(p)+\omega(q)}\Big\}\Big[X_1(p,q) - \frac{X_0(p)}{\omega(p)} X_1^\dagger(p,q) \frac{X_0(q)}{\omega(q)}\Big] + ... \tag{15}$$

$$X_0(p) = \frac{i}{a}\gamma_\mu \sin ap_\mu + \frac{r}{a}\sum_\mu (1-\cos ap_\mu) - \frac{1}{a}M_0, \tag{16}$$

$$X_1(q,p) = \int_k (2\pi)^4 \delta(q-p-k) \, \frac{1}{\sqrt{N}} \lambda_\mu(k) \, V_{1\mu}\left(p+\frac{k}{2}\right), \tag{17}$$

$$\begin{aligned} a\omega(p) &= \sqrt{\sin^2(ap_\mu) + \Big(r\sum_\mu (1-\cos(ap_\mu)) - M_0\Big)^2}, \\ ab(p) &= r\sum_\mu (1-\cos(ap_\mu)) - M_0. \end{aligned} \tag{18}$$

The vertex function is explicitly given as

$$\begin{aligned} V_{1\mu}\left(p+\frac{k}{2}\right) &= i\gamma_\mu \cos a\left(p_\mu + \frac{k_\mu}{2}\right) + r\sin a\left(p_\mu+\frac{k_\mu}{2}\right) \\ &= \frac{\partial}{\partial p_\mu} X_0\left(p+\frac{k}{2}\right). \end{aligned} \tag{19}$$

From (14), the tree level propagator is obtained as

$$\begin{aligned} D_{M(0)}^{-1} &= \frac{a\{b(p)+(1-Ma)\omega(p)\} - i\gamma_\mu \sin(ap_\mu)}{\omega(p)\{1+(1-Ma)^2\} + 2b(p)(1-Ma)\}} \\ &\equiv \frac{A_\mu(p)\gamma_\mu + B(p)}{J(p)}. \end{aligned} \tag{20}$$

From (10) and (13),

$$\begin{aligned} \frac{M_s}{g_v} &= -\int_k \mathrm{Tr}\Big(D_{M(0)}^{-1}(k)\Big) \\ &= -\int_k \frac{2a\{b(k)+(1-Ma)\omega(k)\}}{\omega(k)\{1+(1-Ma)^2\}+2b(k)(1-Ma)}. \end{aligned} \tag{21}$$

Effective action of φ^i, φ_5^i and the gauge field $\lambda_\mu(n)$ is obtained by integrating out the fermions,

$$e^{-S_{eff}} = \int [D\bar{\psi} D\psi] e^{-S}. \tag{22}$$

Especially we are interested in φ_5 and λ_μ part of the effective action, because φ_5 is the quasi-NG boson(pion) and its coupling with the gauge boson is related with anomaly. We define

$$S^{(2)}_{eff}[\varphi_5] = \int_k \frac{1}{2} \varphi_5^i(-k) \Gamma^5_{ij}(k^2) \varphi_5^j(k) \tag{23}$$

where

$$\begin{aligned} \Gamma^5_{ij}(k^2) &= \delta_{ij}\Big[\frac{1}{g_v} + \int_k \mathrm{Tr}[\gamma_5 \langle \psi(k-p)\bar{\psi}(k-p)\rangle \gamma_5 \langle \psi(k)\bar{\psi}(k)\rangle]\Big] \\ &= \delta_{ij}\Big[\epsilon + 2k^2 M_0^2 A(k^2; M)\Big]. \end{aligned} \tag{24}$$

Parameter ϵ in Γ^5_{ij} (24) is proportional to the pion mass and measures the derivation from the limit of the exact chiral symmetry.

In the leading order of the $1/N$,

$$\begin{aligned} \epsilon &= -\frac{2}{a^2} \int_k^\pi \frac{1}{M_s[\omega(k/a)\{1+(1-Ma)^2\} + 2b(k/a)(1-Ma)]^2} \\ &\quad \times \Big[a\{b(k/a) + (1-Ma)\omega(k/a)\}\{\omega(k/a)(1+(1-Ma)^2) + 2b(k/a)(1-Ma)\} \\ &\quad + M_s\{\sin^2 k_\mu + a^2(b(k/a) + (1-Ma)\omega(k/a))^2\}\Big] \\ &= \frac{M_B M_0^2}{M_s}\Big[-\ln(M_0 M^2 a) + \mathrm{const.}\Big] + O(a), \end{aligned} \tag{25}$$

where $\int_k^\pi = \int_{-\pi}^\pi \frac{d^2k}{(2\pi)^2}$ and we took the continuum limit to obtain the last line of (25). From (25), $\epsilon \propto M_B + O(a)$ and therefore the limit $M_B \to 0$ is considered as the chiral limit. It is instructive to compare the above result with that in the continuum theory and the lattice model with the Wilson fermions. The corresponding expression of ϵ in the continuum theory is given as

$$\Big[\epsilon\Big]_{cont} = \frac{2M_B}{M_s} \frac{1}{4\pi} \ln\Big(\frac{\Lambda^2}{M^2}\Big) + O(1/\Lambda), \tag{26}$$

where Λ is the momentum cutoff. It is obvious that ϵ in the overlap formalism has a very close resemblance to that in the continuum theory. On the other hand, the

corresponding expression in the Wilson fermion formalism was obtained in Ref.[5] as follows,

$$
\begin{aligned}
\Big[\epsilon\Big]_W &= -\frac{4r_W}{M_s a}L(r_W) + \frac{2M_B}{M_s}\int_k^\pi \frac{1}{I(k)}, \\
I(k) &= \sum_\mu \sin^2 k_\mu + \Big(-2r_W\sum\sin^2\frac{k_\mu}{2} + Ma\Big)^2, \\
L(r_W) &= \int_k^\pi \frac{\sum\sin^2(k_\mu/2)}{I(k)},
\end{aligned}
\tag{27}
$$

where r_W is the Wilson parameter. Therefore it is obvious that the fine tuning of the "bare mass" M_B and the Wilson parameter r_W is required in order to reach the chiral limit. In this sense, the overlap fermion is better than the Wilson fermion.

It is also straightforward to calculate $A(k^2;M)$ in (24),

$$
\begin{aligned}
A(k^2;M) &= \frac{1}{4\pi\sqrt{k^2(k^2+\mu^2)}}\ln\frac{k^2+2\mu^2+\sqrt{(k^2+2\mu^2)^2-4\mu^4}}{k^2+2\mu^2-\sqrt{(k^2+2\mu^2)^2-4\mu^4}} \\
&\to \frac{1}{4\pi\mu^2}+(k^2).
\end{aligned}
\tag{28}
$$

where $\mu = M_0 M$ and therefore the pion mass is given as $m_\pi^2 = 2\pi M^2\epsilon$.

There exists a mixing term of the gauge boson λ_μ and the pion φ_5^0,

$$
S_{eff}^{(2)}[\lambda_\mu,\varphi_5^0] = -2\sqrt{L}M_0^2 M\int_k\sum\lambda_\mu(-k)\epsilon_{\mu\nu}k_\nu A(k^2;M)\varphi_5^0(k), \tag{29}
$$

which is identical with the continuum calculation. This mixing term is related to the discussion of the U(1) problem in QCD_4 and the above result suggests that the correct anomaly appears in the Ward-Takahashi identity of the axial-vector current.

We shall examine the PCAC relation. By changing variables as follows in the path-integral representation of the partition function[2],

$$
\begin{aligned}
&\psi(n)\to\psi(n)+\tau^k\theta^k(n)\gamma_5\Big\{\delta_{nm}-aD(n,m)\Big\}\psi(m), \\
&\bar\psi(n)\to\bar\psi(n)\left\{1+\tau^k\theta^k(n)\gamma_5\right\},
\end{aligned}
$$

[2]We employ this form of change of variables instead of that is given by (6). This is merely for technical reason here.

$$\phi^i(n) \to \phi^i(n) + d^{ikj}\theta^k \phi_5^j(n),$$
$$\phi_5^i(n) \to \phi_5^i(n) - d^{ikj}\theta^k \phi^j(n), \tag{30}$$

we obtain the Ward-Takahashi(WT) identity,

$$\begin{aligned}\langle \partial_\mu j^k_{5,\mu}(n) - 2M(\bar{\psi}\tau^k\gamma_5\psi)(n) + \frac{2\sqrt{N}}{g_v} M_s \varphi_5^k(n) \\ + D_A^k(n) - \delta^{k0} N\sqrt{L} a \mathrm{Tr}[\gamma_5 D(n,n)]\rangle = 0,\end{aligned} \tag{31}$$

where the last term comes from the measure of the path integral, and the explicit form of the current operator $j^k_{5,\mu}$ is obtained by Kikukawa and Yamada[9] as follows,

$$j^k_{5,\mu}(n) = \tau^k \sum_{lm} \bar{\psi}(l) K^5_{n\mu}(l,m)\psi(m), \tag{32}$$

$$K^5_{n\mu}(l,m) = \left\{ K_{n\mu} \frac{H}{\sqrt{H^2}} \right\}(l,m), \tag{33}$$
$$H = -\gamma_5 X, \tag{34}$$

$$aK_{n\mu}(l,m) = \gamma_5 \left\{ \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{(t^2+H^2)} \left(t^2 W_{n\mu} - H W_{n\mu} H \right) \frac{1}{(t^2+H^2)} \right\}_{lm}, \tag{35}$$

$$W_{n\mu}(l,m) = \gamma_5 \left\{ \frac{1}{2}(\gamma_\mu - 1)\delta_{nl}\delta_{n+\hat{\mu},m} U_{n\mu} + \frac{1}{2}(\gamma_\mu + 1)\delta_{l,n+\hat{\mu}}\delta_{nm} U^\dagger_{n+\hat{\mu},\mu} \right\}. \tag{36}$$

and the operator $D_A^k(n)$ is given by

$$\begin{aligned}D_A^k(n) &= aM \sum_m \bar{\psi}(n)\tau^k\gamma_5 D(n,m)\psi(m) \\ &+ \frac{a}{\sqrt{N}} \bar{\psi}(n)\Big(\varphi^i(n)\gamma_5 - \varphi_5^i(n)\Big)\tau^i\tau^k \sum_m D(n,m)\psi(m).\end{aligned} \tag{37}$$

By integrating out the fermions, the above WT identity is expressed in terms of the pions and the gauge field. Matrix element $D_5^A = \langle \varphi_5^k | D_A^k | 0\rangle$ has the contribution from the following two terms,

$$D_5^A(p) = \{D_5^A(p)\}^a + \{D_5^A(p)\}^b,$$
$$\{D_5^A(p)\}^a = \sqrt{N} a \int_q \mathrm{Tr}[\langle \psi(q)\bar{\psi}(q)\rangle D_0(q)] \times \varphi_5^k(p)$$

$$
\begin{aligned}
&= \sqrt{N} a \varphi_5^k(p) \int_q \frac{2(2-Ma)(\omega(q)+b(q))}{J(q)}, \\
\{D_5^A(p)\}^b &= -\sqrt{N} a M \int_q \varphi_5^k(p) \mathrm{Tr}[\langle \psi(p+q)\bar{\psi}(p+q)\rangle \gamma_5 \langle \psi(q)\bar{\psi}(q)\rangle \gamma_5 D_0(p+q)] \\
&= -\sqrt{N} a M \varphi_5^k(p) \int_q \frac{2}{J(q)J(q-p)} \Big[-M \sin aq_\mu \sin a(q-p)_\mu \\
&\quad -(2-Ma)(\omega(q)+b(q)) a \{ b(q-p) + (1-Ma)\omega(q-p) \} \Big]. \qquad (38)
\end{aligned}
$$

From (25) and (38), we obtain

$$
D_5^A(p) = -2\sqrt{N} M_s \epsilon \varphi_5^k(p). \qquad (39)
$$

In a similay way, matrix element $D_\mu^A = \langle \lambda_\mu | D_A^0 | 0 \rangle$ is evaluated as,

$$
\begin{aligned}
D_\mu^A &= \{D_\mu^A(p)\}^a + \{D_\mu^A(p)\}^b, \\
\{D_\mu^A(p)\}^a &= NM\sqrt{L} \int_{qq'} \mathrm{Tr}[\langle \psi(q)\bar{\psi}(q')\rangle \gamma_5 D_0(p+q') \langle \psi(p+q')\bar{\psi}(p+q')\rangle \\
&\quad \times V(p+q', q)] \delta^{k0} \\
\{D_\mu^A(p)\}^b &= -NM\sqrt{L} \int_q \mathrm{Tr}[\langle \psi(q)\bar{\psi}(q)\rangle \gamma_5 \delta^{k0} V(p+q, q)]. \qquad (40)
\end{aligned}
$$

It is not so difficult to show that the above two terms cancel with each other and $D_\mu^A(p) = 0$.

On the other hand, the last term of (31) is evaluated by a similar method for the four-dimensional case by Kikukawa and Yamada[8], and we obtain

$$
a\mathrm{Tr}[\gamma_5 D(n,n)] = \frac{i}{\pi\sqrt{N}} \sum_{\mu\nu} \epsilon_{\mu\nu} \partial_\nu \lambda_\mu. \qquad (41)
$$

Equation (41) is nothing but the chiral anomaly in two dimensions.

Then the final form of the WT identity is given by

$$
\begin{aligned}
\partial_\mu j_{5,\mu}^k &= i\delta^{k0} \frac{\sqrt{NL}}{\pi} \sum_{\mu\nu} \epsilon_{\mu\nu} \partial_\nu \lambda_\mu + 2M\epsilon\sqrt{N} \varphi_5^k \\
&= i\delta^{k0} \frac{\sqrt{NL}}{\pi} \sum_{\mu\nu} \epsilon_{\mu\nu} \partial_\nu \lambda_\mu + \sqrt{\frac{2N}{\pi}} m_\pi^2 \times \frac{\varphi_5^k}{\sqrt{2\pi M^2}}. \qquad (42)
\end{aligned}
$$

Then it is obvious that the PCAC relation is satisfied in the overlap fermion formalism.

In this paper, we studied the overlap fermion formalism by using the two-dimensional gauged Gross-Neveu model in the large-N limit, and showed that the pion mass is automatically proportional to the bare quark mass (i.e., the current quark mass) without any fine tuning and that the PCAC relation is satisfied. This result means that the chiral limit of the overlap fermion formalism is reached by $M_B \to 0$[10]. This is in sharp contrast to the Wilson fermion formalism in which fine tuning of the Wilson parameter is required, and it is expected that the overlap fermion is quite useful for numerical studies of lattice QCD_4 and other realistic theories.

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